Absence of Renormalization of the Specific Heat Coefficient of the Interacting Fermion Systems

Yunkyu Bang*

Department of Physics, Chonnam National University, Kwangju 500-757,
and Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea
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Contrary to the longtime and widely conceived belief, we proved that the specific heat coefficient γ –also called Sommerfeld coefficient – of the interacting Fermion system is not renormalized by the interaction as far as the system remains a Fermi liquid state.

Introduction – Fermi liquid theory[1–3] is the most fundamental conceptual building block of the modern quantum theory of the interacting fermion systems such as metals, semiconductors, superconductors, liquid ^{3}He, neutron stars, etc. In a nutshell, it suggests that an interacting fermion system can be one-to-one mapped to a non-interacting fermion system for the low energy excitations. In the process of this adiabatic mapping, the essential effect of the interaction is to renormalize the original bare fermions into a renormalized fermionic "*quasiparticles*". While the charge and spin quantum numbers – when they exist – of the fermions are protected by the gauge invarience[4] and relativity, the mass of the fermion in the condensed matter is an effective mass from the beginning and can be renormalized to be a different value from the original effective mass m_0 of the non-interacting limit. Therefore, the renormalized effective mass m^* of the quasiparticle (q.p.) is the single most important quantity which determines the low energy properties of the interacting fermion systems. Hence, the reliable measurement of this quantity by experiments should be of principal importance to study the nature and strength of the interaction of the fermionic system.

There are several different probes to measure the effective mass: specific heat (SH) coefficient, de Haas-van Alphen (dHvA) effect, angle resolved photo-emission spectroscopy (ARPES), optical spectroscopy, etc. Although some interpretations might be necessary to extract the value of m^* from the above listed measurements, theoretically all these measurements should give consistently the same information about the effective mass m^*. For example, the ARPES measures the q.p. energy dispersion $E(\mathbf{k})$ *vs* momentum $\mathbf{k}$ and directly shows us, without interpretation, how heavy or light the q.p.s move. The dHvA effect similarly depends on the q.p. dispersion $E(\mathbf{k})$, so that its measurement also provides a direct information of the renormalized mass. However, since the construction of the Landau Fermi liquid phenomenology[1] and its theoretical justifications[2, 3, 5, 6], the most commonly used probe for the effective mass of the q.p.s in the Fermi liquid systems is the measurement of the SH coefficient. In particular, Luttinger had shown in his seminal paper [3] in 1960 that the SH coefficient $\gamma (\equiv \lim_{T\to 0} C(T)/T)$ should be enhanced from the non-interacting value γ_0 such as $\gamma/\gamma_0 = m^*/m_0$. Since then, the measurement of γ has been established as the most important tool to measure the effective mass of the fermionic q.p.s. in the condensed matter systems.

In this paper, we showed that there is an error in the proof of Luttinger and the SH coefficient in the interacting fermion system is not renormalized by the interaction, hence $\gamma/\gamma_0 = 1$ regardless of the strength of the fermion-fermion interaction. Our finding should have far reaching consequences in the study of various interacting fermion systems such as strongly correlated metals, liquid ^{3}He, neutron stars, etc. In this paper, we will be focusing only on the questions of where was wrong in the Luttinger's proof and what is the correct answer for the SH coefficient γ of the interacting fermion systems.

SH coefficient γ and DOS – It is well known that the SH coefficient of the non-interacting fermion system γ_0 is given by

$$\lim_{T\to 0} C(T)/T \equiv \gamma_0 = \frac{\pi^2}{3} N_0(0),\qquad(1)$$

where $N_0(0)$ is the density of states (DOS) of the non-interacting fermion system at the chemical potential. Intuitively, the SH coefficient of the interacting fermion system γ is expected to be given with the above equation by replacing $N_0(0)$ by the DOS of the interacting fermion system $N(0)$ such as $\gamma = \frac{\pi^2}{3} N(0)$. But this absolutely reasonable intuition falls in a serious trouble as follows. The DOS $N(0)$ of the interacting fermion system can be calculated if we know the exact one-particle Green's function which is formally written as $G(k,\omega) = \frac{1}{\omega - \varepsilon(k) - \Sigma(k,\omega)}$ with the exact self-energy $\Sigma(k,\omega)$. And it is almost trivial to show that $N(0) = N_0(0)$, as shown below, which leads us to conclude that $\gamma = \gamma_0$. However, this conclusion is in stark contrast to the common knowledge that the SH coefficient should be enhanced by interaction such as $\gamma/\gamma_0 \approx m^*/m_0 > 1$. There are two possible options to resolve this dilemma: (1) $\gamma = \frac{\pi^2}{3} N(0)$ is not true for the interacting system; or (2) the common belief $\gamma/\gamma_0 \approx m^*/m_0$ is wrong.

Let us begin with showing $N(0) = N_0(0)$. It is well known that $\lim_{T,\omega\to 0} \Sigma(k,\omega) = -\lambda_k \omega - i\delta$ in the Fermi liquid state [2, 3]. Then

$$N(0) \equiv -\frac{1}{\pi} \sum_k Im G(k, \omega = 0) \tag{2}$$

$$= N_0(0) \lim_{\omega \to 0} \int \frac{d\varepsilon}{\pi} \, Im \frac{-1}{[1+\lambda_k]\omega - \varepsilon + i\delta} \tag{3}$$

$$= N_0(0) \lim_{\omega \to 0} \int \frac{d\varepsilon}{Z_k} \, \delta(\omega - \frac{\varepsilon}{Z_k}) \tag{4}$$

$$= N_0(0) \tag{5}$$

where the wave function renormalization parameter $Z_k = 1 + \lambda_k$ ($Z_k > 1$) is defined. Therefore, $N(0) = N_0(0)$ is a rigorous identity as far as the Fermi liquid state holds even with an arbitrary momentum dependent self-energy $\Sigma(k, \omega)$.

To demonstrate the correctness of the result of Eq.(5), we show the numerical results of $N(\omega) = -\frac{1}{\pi} \sum_k Im G(k, \omega)$ of a simple toy model in Fig.1(a) with varying strength of interaction. In this examplary calculations, we assumed a box like DOS for the non-interacting fermion system as $N_0(\omega) = 1.0$ for $-\Lambda < \omega < \Lambda$ and the effect of interaction is simulated by the Fermi liquid type self-energy $Im\Sigma(\omega) = \alpha \omega^2$ for $-\Lambda < \omega < \Lambda$ including the corresponding real part $Re\Sigma(\omega)$. We chose $\Lambda = 5$. The results are self-explaining, showing $N(0) = N_0(0)$ for all interaction strength of α. Increasing the interaction strength, the width of the q.p. DOS around $\omega = 0$ becomes progressively narrowed and the spectral weight outside of it is depleted toward the high energy region which is not fully displayed here but the total spectral weight of the DOS should be conserved. The width of the q.p. DOS around $\omega = 0$ is roughly proportional to $\sim 1/Z$ and the value of Z is determined by the combination of the values of the interaction strength α and the band width scale Λ.

SH coefficient γ of Interacting Fermi Systems – To find an exact theoretic formula to calculate the SH coefficient γ of the interacting Fermi systems, we start with the same Hamiltonian for the interacting fermion system used by Luttinger and Ward [2]

$$H = \sum_r \varepsilon_r c_r^\dagger c_r + \frac{1}{2} \sum_{r,s,r',s'} c_r^\dagger c_s^\dagger c_{r'} c_{s'} (rs|v|r's') \tag{6}$$

where ε_r is the energy measured from the chemical potential of the non-interacting single particle states with the index $r = (k, \sigma)$ for both momentum and spin. $c_r^\dagger, c_r$ are the creation and annihilation fermion operators, respectively, and $(rs|v|r's')$ is the general four point fermion interaction matrix.

In Ref.[2], Luttinger and Ward wrote down the celebrated free energy functional of the interacting fermion system as

$$\Omega(T) = -T \sum_{r,n} e^{i\omega_n 0^+} \{\ln[\varepsilon_r + \Sigma_r(\omega_n) - i\omega_n]$$

$$+ G_r(\omega_n)\Sigma_r(\omega_n)\} + \Omega' \tag{7}$$

where $\omega_n = \pi T(2n+1)$ is Matsubara frequency. $G_r(\omega_n)$ and $\Sigma_r(\omega_n)$ are the full Green's function and the full proper self-energy, respectively. The functional Ω' is defined by LW (refers Ref.[2]) as
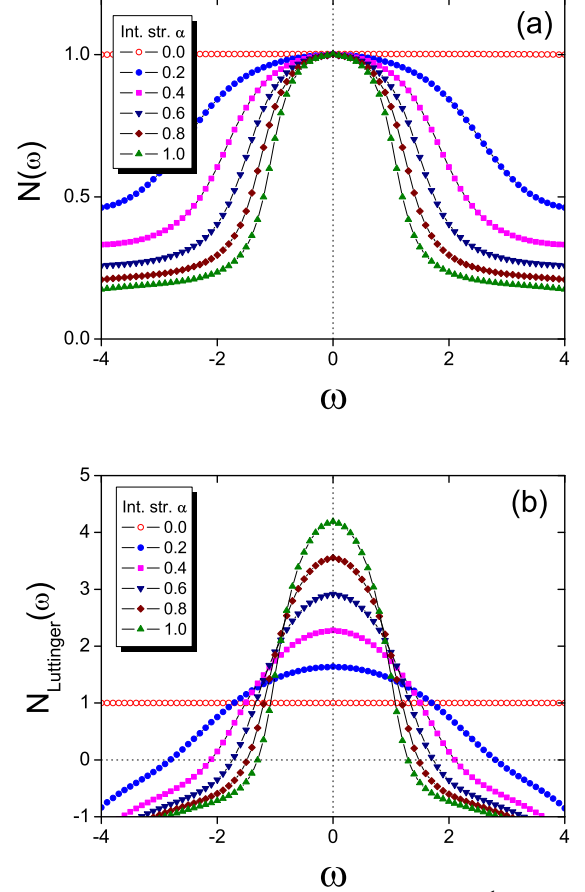

FIG. 1: (Color online) (a) The full DOS $N(\omega) = -\frac{1}{\pi} \sum_k Im G(k, \omega)$ with a phenomenological Fermi liquid type self-energy $Im\Sigma(\omega) = \alpha \omega^2$ with $\alpha = 0, 0.2, 0.4, 0.6, 0.8$ and 1.0, respectively (the K-K related $Re\Sigma(\omega)$ is included). A constant DOS $N_0(\omega) = 1.0$ was chosen for the non-interacting case ($\alpha = 0$). (b) The results of calculations of $-\frac{1}{\pi} \sum_k Im\{G(k, \omega)(1 - \frac{\partial \Sigma}{\partial \omega})\}$ with the same self-energies as in (a).

$$\Omega' = \begin{pmatrix} \text{contribution of all closed-linked skeleton diagrams,} \\ \text{but with replacing all Green's function lines by} \\ \text{the full Green's functions } G_r(\omega_n). \end{pmatrix} \tag{8}$$

The explicit expression of Ω' was given in LW(50) (refers Eq.(50) of Ref.[2]). The functional Ω' was ingeniously designed to satisfy the famous variational theorem of the free energy functional:

$$\frac{\partial \Omega}{\partial \Sigma_r} = 0. \tag{9}$$

And this theorem can be easily proven if the functional Ω' satisfies the following variational property

$$\frac{\partial \Omega'}{\partial \Sigma_r(\omega_n)} = T \sum_{r,n} [G_r(\omega_n)]^2 \Sigma_r(\omega_n) \tag{10}$$

which was shown in LW(51). Using the same reasoning used to derive Eq.(10), we can derive a similar variational relation

$$\frac{\partial \Omega'}{\partial i\omega_n} = -T\sum_{r,n}[G_r(\omega_n)]^2(1-\frac{\partial \Sigma_r}{\partial i\omega_n})\Sigma_r(\omega_n). \quad (11)$$

The above relation will be used for our purpose later and the proof of it is trivial if we note the expression of $G_r^{-1} = i\omega_n - \varepsilon_r - \Sigma_r(\omega_n)$ and replace $\frac{\partial}{\partial \Sigma_r}$ in Eq.(10) by $\frac{\partial}{\partial i\omega_n}$.

In order to calculate the entropy from the free energy functional Eq.(7), we need to extract the leading temperature dependent parts of it. Using a standard method of the Matsubara frequency summation, Eq.(7) is written as

$$\Omega(T) = \oint \frac{dz}{2\pi i}n_F(z)\sum_r\{\ln[\varepsilon_r + \Sigma_r(z) - z]$$
$$+G_r(z)\Sigma_r(z)\} - \oint \frac{dz}{2\pi i}n_F(z)\,\Omega'(z) \quad (12)$$

where all Matsubara frequencies are analytically transformed to complex numbers as $i\omega_n = z$ and the functional $\Omega'(z)$ is also understood as $\Omega'(i\omega_n \to z)$ replacing the overall Matsubara frequency summation $-T\sum_n$ in the original definition of the functional $\Omega'(i\omega_n)$ by the contour integral $\oint \frac{dz}{2\pi i}n_F(z)$ with the Fermi-Dirac distribution function $n_F(z)$. Now it is clear that there are only two places where contain the temperature dependence in the above free energy functional Eq.(12): $n_F(z)$ and $\Sigma_r(z)$. As Luttinger argued [3], the leading temperature dependence should come from the explicit summation of $i\omega_n$ (equivalently in $n_F(z)$) and the temperature variation of "$\Sigma_r(T) - \Sigma_r(T=0)$" is a higher order and should be neglected. Therefore, using $S(T) = -d\Omega(T)/dT$, we can write down $S(T)$ as follows

$$S(T) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi T}\omega\Big[\frac{\partial n_F(\omega)}{\partial \omega}\Big]$$
$$\cdot \sum_r Im\{\ln G_r^{-1}(\omega) + G_r(\omega)\Sigma_r^0(\omega)\}$$
$$- \int_{-\infty}^{\infty}\frac{d\omega}{\pi T}\omega\Big[\frac{\partial n_F(\omega)}{\partial \omega}\Big]Im\Omega'(\omega), \quad (13)$$

where the contour path of $\oint$ is deformed along the real frequency axis à la the appendix of Ref.[2]; the ω integration for $[-\infty,\infty]$ should be carried infinitesimally above the real axis, i.e. for $\omega + i\eta$. $\Sigma_r^0(\omega)$ means $\Sigma_r(\omega,T=0)$ and it is understood that every $\Sigma_r(\omega,T)$ implicit in the above expression is replaced by $\Sigma_r^0(\omega)$. While the above equation for $S(T)$ is the undoubtedly correct expression, Luttinger argued in Ref.[3] that the leading temperature dependence of $\Omega(T)$ (Eq.(7)) is contained only in

$$\Omega_{temp}(T) \approx -T\sum_{r,n}e^{i\omega_n 0^+}\ln[\varepsilon_r + \Sigma_r(\omega_n) - i\omega_n] \quad (14)$$

because the leading temperature dependent parts in the remaining terms $-T\sum_{r,n}e^{i\omega_n 0^+}[G_r(\omega_n)\Sigma_r(\omega_n)]+\Omega'$ cancels exactly each other. Hence, $S(T)$ considered by Luttinger is the following

$$S_{Luttinger}(T) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi T}\omega\Big[\frac{\partial n_F(\omega)}{\partial \omega}\Big]\sum_r Im\{\ln[\varepsilon_r + \Sigma_r^0 - \omega]\}$$
$$= \int_{-\infty}^{\infty}\frac{d\omega}{\pi T}\omega\Big[\frac{\partial n_F(\omega)}{\partial \omega}\Big]\sum_r Im\{\ln G_r^{-1}(\omega)\}. \quad (15)$$

Note the difference of the expressions of the entropy $S(T)$ (Eq.(13)) derived in this paper and the one $S_{Luttinger}(T)$ (Eq.(15)) considered by Luttinger in Ref.[3]. Below we will show that the exact cancellation of the remaining terms – claimed by Luttinger – is incomplete and therefore we have to keep the full expression of the entropy Eq.(13). Expectedly the calculation results of the SH coefficient γ from Eq.(13) and $\gamma_{Luttinger}$ from Eq.(15) are totally different: the former one gives γ unrenormalized regardless of the strength of the interaction while the latter one gives an enhanced $\gamma_{Luttinger}$ proportional to the interaction strength as widely believed in the community ever since the proof of Luttinger [3].

To obtain $\gamma \equiv \lim_{T\to 0}C(T)/T = \lim_{T\to 0}dS(T)/dT$, we only need to extract T-linear terms in $S(T)$ or $S_{Luttinger}(T)$. Utilizing Sommerfeld expansion, we then only need to extract ω-linear terms in the integrand of $Im...$ in $S(T)$ or $S_{Luttinger}(T)$. Let us first calculate $\gamma_{Luttinger}$ from $S_{Luttinger}$. The leading Taylor expansion of the integrand in $S_{Luttinger}$ is

$$Im\{\ln[\varepsilon_r + \Sigma_r^0 - \omega]\} = ImG_r(\omega=0)(1-\frac{\partial Re\Sigma_r^0}{\partial \omega}\Big|_{\omega=0})\omega$$
$$+ O(\omega^2)... \quad (16)$$

Using the notation $(1-\frac{\partial Re\Sigma_r^0}{\partial \omega}\Big|_{\omega=0}) = 1+\lambda_r = Z_r$, we obtained

$$-\frac{1}{\pi}\sum_r ImG_r(\omega=0)(1+\lambda_r) \approx Z\cdot N(0) = Z\cdot N_0(0) \quad (17)$$

where Z is a Fermi surface average of Z_r. In Fig.1(b), we showed the numerical calculations of $-\frac{1}{\pi}\sum_r Im\{G_r(\omega)(1-\frac{\partial \Sigma_r^0}{\partial \omega})\}$ with the varying interaction strength. This quantity has no direct physical meaning (it becomes even negative at higher energies) but its zero frequency value $N_{Luttinger}(\omega=0)$ clearly demonstrated the result of Eq.(17) and showed what quantity was used by Luttinger for the calculation of the SH coefficient. Substituting the result of Eq.(17) into Eq.(15), we can derive the same result as Luttinger had obtained[3] as

$$\gamma_{Luttinger} = \frac{\pi^2}{3}Z\cdot N_0(0), \quad (18)$$

so that the SH coefficient $\gamma_{Luttinger}$ is indeed enhanced by the factor Z compared to the non-interacting case.

Now let us use the exact expression $S(T)$ of Eq.(13) to derive γ. The coefficients of the ω-linear terms of the integrand

of Eq.(13), $\{\ln G_r^{-1}(\omega) + G_r(\omega)\Sigma_r^0(\omega)\} - \Omega'(\omega)$, are the following

$$
\begin{aligned}
&= G_r(1 - \frac{\partial \Sigma_r^0}{\partial \omega}) \\
&- [G_r]^2 \Sigma_r^0 (1 - \frac{\partial \Sigma_r^0}{\partial \omega}) + G_r \frac{\partial \Sigma_r^0}{\partial \omega} \\
&+ [G_r]^2 \Sigma_r^0 (1 - \frac{\partial \Sigma_r^0}{\partial \omega}) \\
&= G_r.
\end{aligned} \tag{19}
$$

Above we have arranged the expansions of each three terms $\ln G_r^{-1}(\omega)$, $G_r(\omega)\Sigma_r^0(\omega)$ and $-\Omega'(\omega)$ into three separate lines for clarity. There are lots of cancellations and the final result should be compared to Eq.(16) obtained from $S_{Luttinger}(T)$ Eq.(15). In fact, the above cancellation is the consistent result of the variational theorem of Eq.(9) which requires that all variations of $\partial \Sigma_r$ in the total free energy functional Ω should sum up to zero [2]. In this sense, the expression of $S_{Luttinger}$ in Eq.(15) with Eq.(16) cannot be correct since it contains $\partial \Sigma_r$ term.

Now it is a trivial matter to calculate the SH coefficient γ substituting the result of Eq.(19) into Eq.(13) as

$$
\begin{aligned}
\gamma &= \frac{\pi^2}{6} \sum_r \frac{-ImG_r(\omega = 0)}{\pi} \\
&= \frac{\pi^2}{3} \cdot N(0) = \frac{\pi^2}{3} \cdot N_0(0)
\end{aligned} \tag{20}
$$

where we have counted the spin summation factor 2 in the index $r = (k, \sigma)$. The above result shows that the SH coefficient γ of the interacting Fermi system is proportional to the interacting DOS at the chemical potential $N(0)$, which is consistent with our physical intuition. However, due to the absence of renormalization of the DOS at chemical potential, i.e., $N(0) = N_0(0)$, which we have shown in Eq.(5), the SH coefficient γ is not renormalized by the interaction as far as the system remains a Fermi liquid state.

Other Physical Quantities – The renormalized q.p. mass $m^* \approx Z m_0$ due to interaction is measured by different experimental probes. Indeed the energy dispersion of the q.p. pole $E(k)$, defined by $\omega - \varepsilon(k) - \Sigma(k, \omega) = 0$, is renormalized as $E(k) \approx \varepsilon(k)/Z$ and should be directly measured by ARPES without any interpretation or confusion. Another common tool to measure m^* is the dHvA effect with the applied external field H. In this case, the effective mass is measured from the temperature reduction factor of the signal strength which is given by the Lifshitz-Kosevich formula $R_T \sim \exp(-T/\omega_c)$ [7], where ω_c is the cyclotron frequency. ω_c is determined by the q.p. energy distance between the Landau levels quantized by the field H as $\Delta E = \hbar \omega_c$, and the Landau level is determined by the q.p. dispersion $E(k)$ to the first approximation, hence $\omega_c = eH/m^*c$. Therefore, the dHvA effect measurement can provide an information of m^*. Lastly, the optical spectroscopy measurements need a more careful interpretation. The total spectral density near the Fermi level is not enhanced by interaction as shown in Eq.(5), however the width of the q.p. dispersion is narrowed by the factor $1/Z$ as shown in Fig.1(a). Therefore, for example, the width of the Drude spectra in the optical conductivity should be reduced by the factor $1/Z$, while the absolute magnitude of the zero frequency conductivity $\sigma(\omega = 0)$ nor the total Drude spectral weight is not expected to be enhanced. However, because the optical conductivity is a transport property, it is necessary also to count on the scattering rate due to the interaction besides the q.p. DOS. Therefore, for more complete details of the optical properties of the interacting fermion systems, we need to analyze the two particle correlation function which was beyond the scope of the current paper.

Conclusions – In this paper, we have proven that the SH coefficient of the interacting fermion system is not renormalized by the interaction and remains basically the same value as the non-interacting limit of the system. This result is in stark contrast to the longtime accepted idea of the interaction-enhanced SH coefficient since the proof of Luttinger in 1960 [3]. The implications of our finding should be far reaching and profound because the enhanced SH coefficient γ in the interacting Fermion systems has been accepted and utilized for the last 50 years as a pivotal building concept in the study of the interacting Fermi liquid systems both in theory and in experiment. We need to rethink many of the previous ideas and measurements based on this *–now proven wrong –* concept.

* ykbang@chonnam.ac.kr
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